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Firmoviscous and Anelastic Properties of Fluids and Their Effects on the Propagation of Compression Waves

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In any real fluid, the propagation of acoustic waves will always be accomplished by a dispersion or scattering action, resulting in part from the astatic nature of the compressive modulus. While the resulting significant dissipation of available energy can be related readily to simple dynamic properties of the fluid substance, lack of appreciation of these properties has retarded the accumulation of adequate experimental data.

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Concept of a Dynamic Bulk Modulus

In almost all dynamical treatments of fluid flow, a static relationship between the fluid density and the fluid pressure is assumed. For liquids, this relationship is usually expressed in terms of the fluid bulk modulus, K, as manifested by:

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right) dP = \frac{\rho}{K} dP \tag{1}$$

where

ρ = fluid density, lb-sec^2/ft^4
P = fluid pressure, psf
K = fluid bulk modulus, psf

On this relationship nearly all classical developments for wave propagation in slightly compressible fluids have been based. It is the primary aim of this paper to assume a more general dynamical relationship between pressure and density and then cursorily to examine the resulting effects on fluid phenomena such as acoustic wave propagation.

From a general macroscopic and phenomenological standpoint, it is not a priori reasonable to assume such a purely static relationship between pressure changes ΔP, and density changes, $\Delta\rho$, of a fluid under dynamic conditions. Rather, it is more reasonable initially to assume a more complete (albeit linear) form such as

$$a_0\Delta\rho + a_1\Delta\dot{\rho} + a_2\Delta\ddot{\rho} + \cdots = b_0\Delta P + b_1\Delta\dot{P} + b_2\Delta\ddot{P} + \cdots \tag{2}$$

where the dots indicate time derivatives and the coefficients are in general undertermined. This relationship also may be indicated in the equivalent operational forms:

$$\left\{\begin{array}{c} \left(\sum_{k=0}^{\infty} a_k D^k\right)\Delta\rho = \left(\sum_{k=0}^{\infty} b_k D^k\right)\Delta P \\ \text{or } \Delta\rho = \left[\frac{\Sigma b_k D^k}{\Sigma a_k D^k}\right]\Delta P \end{array}\right\} \tag{3}$$

with D = d/dt denoting the differential operator.

Historically, only the very simplest of the infinitude of possibilities for Equation (2) have been considered analytically or have been employed as descriptive models for reducing experimental data. However, for a long period now, it has been recognized how arbitrary these idealizations have been; such awareness has become particularly acute since the advent of generalized continuum mechanics and rheology (1, 2, 3).[1]

We shall here consider only the next approximation beyond the classical; namely, that with a_0, a_1, b_0, b_1 non-zero, and with all the rest of the a_k and b_k assumed vanishingly small. The coefficient a_1 is directly related to what has been termed the second coefficient of viscosity (4, 5)

There is ample physical evidence to assume that there exists a time lag between the application of pressure and the resulting change in density. Using finite incremental changes this phenomenon might be expressed mathematically in the assumed form:

$$\frac{\Delta\rho}{\rho} = \left[\frac{1}{K_1} + \frac{1/K_2}{\tau D+1}\right]\Delta P = \frac{1}{K}\left[\frac{\alpha\tau D+1}{\tau D+1}\right]\Delta P \tag{4}$$

where

τ = retardation time, sec
D = d/dt = differential operation, sec^{-1}
$1/K = (1/K_1) + (1/K_2)$ reciprocal bulk modulus sq ft per lb
$\alpha = (1/K_1)/(1/K)$ a nondimensional ratio, generally less than unity

Consequently a conjectural dynamic or anelastic bulk modulus (which is identical to the static bulk modulus in the steady state) can thus be defined:

$$\mathcal{K} \equiv K\left[\frac{\tau D+1}{\alpha\tau D+1}\right] \tag{5}$$

To illustrate the physical significance of the dynamic bulk modulus, Fig. 1 demonstrates how the density, ρ, or the pressure, P, of a fluid would vary with time due to the application of step change in either quantity.

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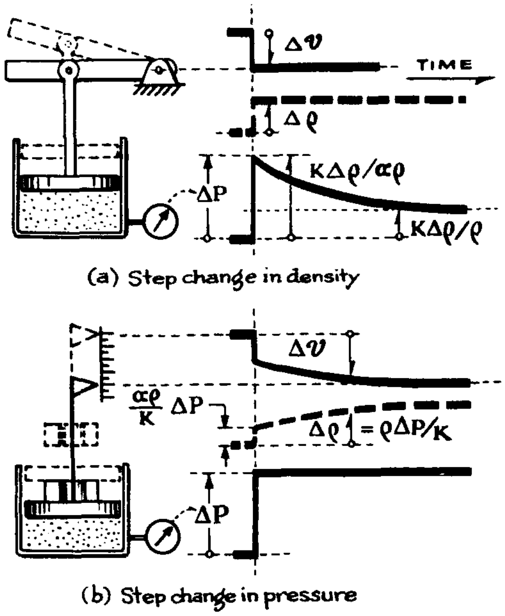
[1] Underlined numbers in parentheses designate References at end of paper.



(a) Step change in density

(b) Step change in pressure

Fig.1 Illustration of the dynamic bulk modulus.

Effect on Wave Equations:

The concept of a dynamic bulk modulus can now be utilized to modify the evaluation of effective fluid capacitance in the derivation of the wave equations. For the purpose of comparison between the pure wave propagation along a pipe and the dispersive propagation involving a dynamic modulus, corresponding variables, equations, and model analogies for the two types will be carried in a parallel fashion as follows:

Nondispersive

Momentum: $$-\frac{\partial P}{\partial s} = I\frac{\partial W}{\partial t} \quad (6a)$$

Elasticity: $$-\frac{\partial W}{\partial s} = C\frac{\partial P}{\partial t} \quad (7a)$$

Dispersive

$$-\frac{\partial P}{\partial s} = \frac{\partial W}{\partial t} \quad (6b)$$

$$-\frac{\partial W}{\partial s} = C\left[\frac{\alpha\tau D+1}{\tau D+1}\right]\frac{\partial P}{\partial t} \quad (7b)$$

where, in addition to the variables previously defined:

s = distance along the pipe, ft
W = weight rate of flow, lb per sec
I = fluid inertance per ft of pipe, sec^2/ft^3

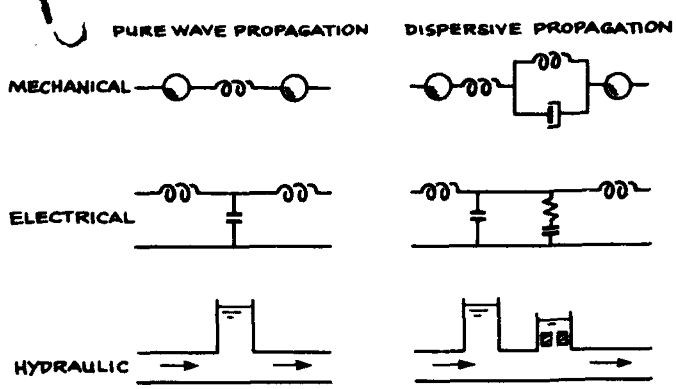


Fig.2 Model analogies for a fluid element having a dynamic bulk modulus.

C = fluid capacitance per ft of pipe, ft
t = time, sec

Equations (6) and (7) can be written in a more general form in terms of an impedance and admittance operator that relate the relationships between pressure and flow as stated in Equations (8) and (9) for which Equations (6) and (7) are merely special forms.

$$-\frac{\partial P}{\partial s} = Z(D)\cdot W \quad (8)$$

$$-\frac{\partial W}{\partial s} = Y(D)\cdot P \quad (9)$$

Having written these expressions, the propagation operator and the characteristic impedance Z_0 for a fluid column can now be derived for the dispersive case in a similar manner to what is usually done for the lossless case. Again for the purpose of comparison, corresponding relationships will be written for the two cases:

Pure Wave Propagation

Impedance per unit length $$Z(D) = ID \quad (10a)$$

Admittance per unit length $$Y(D) = CD \quad (11a)$$

Propagation Operator $$\Gamma = \ell\sqrt{ZY} = \ell\sqrt{IC}\,D = TD = \Gamma^* \quad (12a)$$

Characteristic Impedance $$Z_0 = \sqrt{Z/Y} = \sqrt{I/C} = Z_0^* \quad (13a)$$

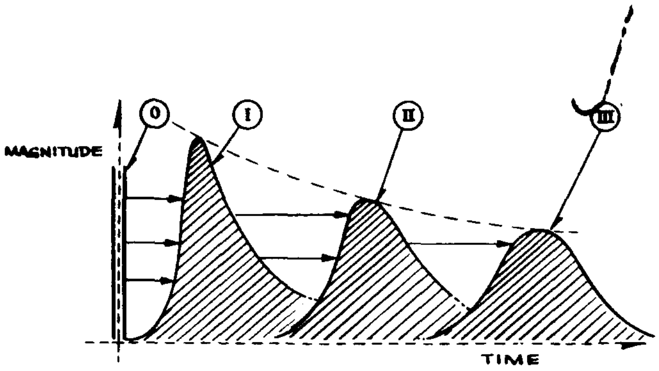


Fig.3 Impulse response for an infinite

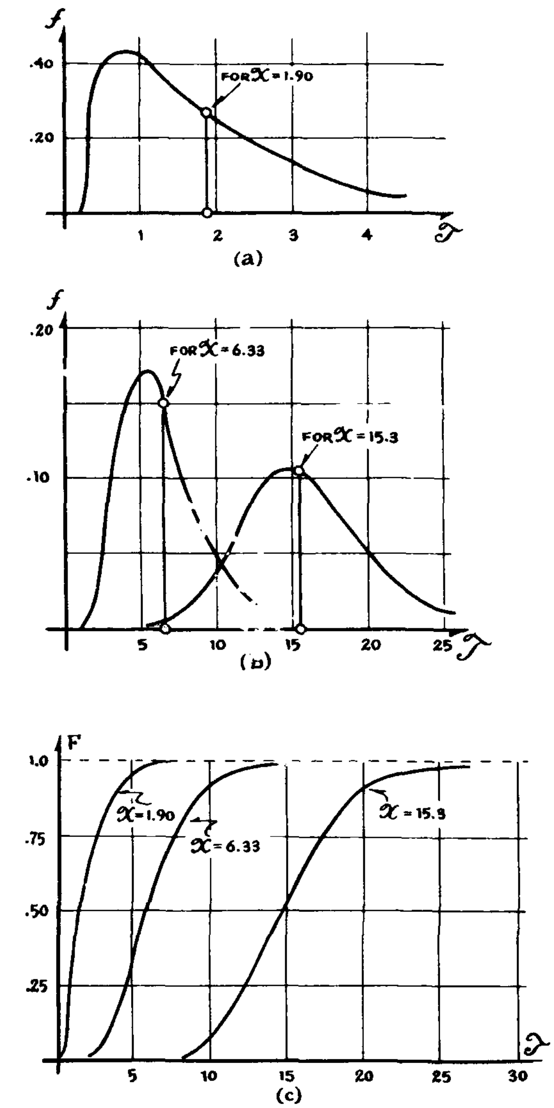


Fig.4 Calculated response curves. (a) Impulse response $\mathcal{X}$= 1.90; (b) Impulse response $\mathcal{X}$ = 6.33, 15.3; (c) step responses.

Dispersive Propagation

$$Z(D) = lD \qquad (10b)$$

$$Y(D) = C\left[\frac{\alpha\tau D+1}{\tau D+1}\right]D \qquad (11b)$$

$$\begin{aligned}\Gamma &= \ell\sqrt{ZY}\\ &= \ell\sqrt{lC}\sqrt{\frac{\alpha\tau D+1}{\tau D+1}}\,D\\ &= \Gamma^k\sqrt{\frac{\alpha\tau D+1}{\tau D+1}}\end{aligned} \qquad (12b)$$

$$\begin{aligned}Z_0 &= \sqrt{Z/Y} = \sqrt{\frac{l}{C}}\sqrt{\frac{\tau D+1}{\alpha\tau D+1}}\\ &= Z_0^*\sqrt{\frac{\tau D+1}{\alpha\tau D+1}}\end{aligned} \qquad (13b)$$

where ℓ = total column length, ft; T = wave travel time along the column, sec. The transfer operator for a pipe also may be written in a generalized matrix form:

$$\underbrace{\begin{bmatrix} P_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma & Z_0\sinh\Gamma \\ \frac{1}{Z_0}\sinh\Gamma & \cosh\Gamma \end{bmatrix} \cdot \begin{bmatrix} P_2 \\ W_2 \end{bmatrix}} \qquad (14)$$

where subscripts 1 and 2 denote upstream and downstream conditions, respectively.

The solution to Relation (14) may be obtained for a given set of conditions and fluid terminations. However, because of the complexity of Γ and Z_0 the general solution cannot be expressed in a simple closed form as in the case of pure wave propagation. There are several methods however for approximating the solution for a specific set of conditions.

Propagation of a Pressure Impulse in a Semi-Infinite Column

For example, if a pressure impulse is applied at one end of a semi-infinite fluid column, the shape of this impulse at any point can be approximated by certain distribution functions as it travels along the column.

Since if such a semi-infinite column the pressure at any point is given by:

$$P(s,t) = \epsilon^{-\Gamma} \cdot P(o,t) \qquad (15)$$

This result follows from Equations (12) with

$$Z_0 W(s,t) \equiv P(s,t) \qquad (16)$$

which is true for the case in question. Following the methods of one of the authors (6, 7), the operator $\epsilon^{-\Gamma}$ may be expanded as follows:

$$\epsilon^{-\Gamma} = \epsilon^{-TD\sqrt{\frac{\alpha\tau D+1}{\tau D+1}}} \tag{17}$$

$$= \epsilon^{-TD+\left(\frac{1-\alpha}{2}\right)T\tau D^2-\cdots} \tag{18}$$

$$= \epsilon^{-T_m D+\frac{1}{2}T_s^2 D^2-\cdots} \tag{19}$$

In Equation (17) the time constants have the subsequent significance

$T_m = T$ = mean delay, sec

$T_s = \left[(1-\alpha)T\tau\right]^{1/2}$ = dispersion time, sec

Hanin (7) and Vaughn (8) have demonstrated for the case $\alpha = 0$ that an impulse at the origin gets increasingly dispersed and attenuated as it propagates. Following suggestions of the present authors, Vaughn showed that its shape can be predicted accurately by the delayed chi-square distribution function (II) and that the limiting form for very early development can be predicted by the complementary error function (I). For large intervals of time the pulse approaches the normal distribution function (III).

Figs. 3 and 4 illustrate these shapes graphically, while the Appendix presents the corresponding mathematical formulas, for the impulse and step response of a semi-infinite fluid medium.

However, the two derived constants, T_m and T_s, adequately define behavior for all types of disturbances and boundary conditions. This fact permits determination of the fluid properties from any adequate and consistent observational data.

Conclusions

All experimental data known to the authors reinforce the belief that such a compressive model as that proposed in the foregoing is of the minimum complexity which will still explain the scattering action readily apparent in observed behavior of fluids. Thus we strongly urge that a concerted effort be made to reduce such observations to the point where the anelastic properties of the more common fluids can be established firmly.

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APPENDIX

Derivation of Dynamic Response Equations

Purely Firmoviscous Propagation ($\alpha = 0$).

For this case it is possible to simplify analysis by first normalizing, or nondimensionalizing, all variables and parameters through introducing into Equation (15) the substitutions:

$$\begin{aligned} &\text{NORMALIZED DISTANCE:} && \mathcal{X} \equiv (T/\tau)\cdot x \\ &\text{NORMALIZED TIME:} && \mathcal{T} \equiv (1/\tau)\cdot t \\ &\text{NORMALIZED DERIVATIVE:} && \mathcal{D} \equiv (\tau)\cdot D \end{aligned} \tag{A}$$

Thus we may now treat the generalized operator:

$$\epsilon^{-\Gamma} \equiv \mathbb{F}(\mathcal{X},\mathcal{D}) = \epsilon^{-\mathcal{X}\mathcal{D}/\sqrt{1+\mathcal{D}}}$$

The condition $\mathcal{D} \to \infty$ corresponds to the instantaneous response, $t \to 0$, and yields:

$$\lim_{\mathcal{D}\to\infty} \mathbb{F}(\mathcal{X},\mathcal{D}) = \mathbb{F}_\infty(\mathcal{X}) = \epsilon^{-\mathcal{X}\sqrt{\mathcal{D}}} \tag{B}$$

This operator applied to a unit impulse or unit step results in the consequent impulse response and step response, in the form:

$$\text{IMPULSE RESPONSE:} \quad f_\infty(\mathcal{X},\mathcal{T}) = \frac{\mathcal{X}}{2\mathcal{T}\sqrt{\pi\mathcal{T}}}\,\epsilon^{-\mathcal{X}^2/4\mathcal{T}} \tag{C}$$

$$\text{STEP RESPONSE:} \quad \mathbb{F}_\infty(\mathcal{X},\mathcal{T}) = 1 - erf\left(\mathcal{X}/\sqrt{2\mathcal{T}}\right)$$

Tables of the functions indicated are readily available in standard handbooks, such as that of Burington and May (10).

For the opposite conditions, when the disturbance has traveled a great distance from the origin, then $t \to \infty$, $\mathcal{D} \to 0$, and we may consider the limiting operator:

$$\lim_{\mathcal{D}\to 0} F(\mathcal{X},\mathcal{D}) \equiv F_0(\mathcal{X}) = \epsilon^{-\mathcal{X}\mathcal{D} + \frac{\mathcal{X}}{2}\mathcal{D}^2} \qquad \text{(D)}$$

corresponding to the impulse and step responses:

$$\text{IMPULSE RESPONSE: } f_0(\mathcal{X},\mathcal{T}) = \frac{1}{\sqrt{2\pi\mathcal{X}}}\,\epsilon^{-\frac{(\mathcal{T}-\mathcal{X})^2}{2\mathcal{X}}} \qquad \text{(E)}$$

$$\text{STEP RESPONSE: } F_0(\mathcal{X},\mathcal{T}) = \int_0^{\mathcal{T}} f_0\,dt = \Phi(\mathcal{T})$$

Thus these results correspond to the well-known Gaussian distribution or normal probability distribution, with mean = variance = $\mathcal{X}$. This function is tabulated in many handbooks, such as that previously cited (10).

Between these two extremes many models would be possible. That used by Vaughan (9), as mentioned earlier, is a "delayed chi-square distribution," which is an approximation in the form:

$$F_{1/m}(\mathcal{X},\mathcal{D}) \cong \epsilon^{-T_1\mathcal{D}}/(1+T_2\mathcal{D})^{m/2} \qquad \text{(F)}$$

with T_1, T_2, normalized time constants and m the number of degrees of freedom of the chi-square distribution. An excellent fit is obtained by matching the first three impulse moments, which yields the conditions:

$$\begin{aligned} T_1 &= (1/9)\,\mathcal{X} = 0.111\,\mathcal{X} \\ T_2 &= 9/8 = 1.125 \\ m &= (128/81)\,\mathcal{X} = 1.580\,\mathcal{X} \end{aligned} \qquad \text{(G)}$$

The corresponding values of impulse response, $f_{1/m}$, and step response $F_{1/m}$, are readily found from any of the available tables of the chi-square distribution such as those previously cited (10).